NORMAL FORMS, INVARIANTS, AND BIFURCATIONS OF NONLINEAR CONTROL SYSTEMS

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Abstract. This paper addresses the problem of bifurcation analysis for nonlinear control systems. It is a review of some results published in recent years on bifurcations of control systems based on the normal form approach. We address three problems for systems with a single uncontrollable mode, namely the problem of normal forms, bifurcations of control systems, and bifurcation control by state feedback.

1 Introduction

Nonlinear dynamical systems exhibit complicated performance around bifurcation points. As the parameter of a system is varied, changes may occur in the qualitative structure of its solutions around a point of bifurcation. This paper addresses the problem of bifurcation analysis for nonlinear control systems. This is a review paper that summarizes the main results published in recent years on bifurcations of control systems based on normal forms and invariants. More specifically, we focus on the results from [15], [17], [18], [19] and [20].

The theoretical research on the bifurcation of control systems is a relatively young subject. Nevertheless, it has attracted increasing attention. For example, in [1], [7], [27], [33], and other related publications, bifurcation control based on the projection method was developed with some interesting applications. Many other theoretical approaches or engineering applications of bifurcation control can be found in the literature (e.g., [6], [10], [12], and [27]). A review of bifurcation and chaos in control systems can be found in [6]. More references on related topics can be found in [5], a bibliography of publications on bifurcation and chaos in control systems. Given the large volume of new developments in this field, it is impossible to review all the methods used for bifurcation control. Instead, this paper focuses on the approach of normal forms and invariants only.

The theory of bifurcation control based on normal forms was developed in mainly three parts, the normal form and invariants, the bifurcation of equilibrium sets, and bifurcation control by feedback. These results were published separately in several papers for the past ten years ([15], [17], [18], [19] and
This review is to summarize these integral parts in one article so that the connections between the separately published papers and the philosophy of the overall approach can be illustrated efficiently. The paper is organized as follows. Several bifurcation problems are formulated in §2 for nonlinear control systems. Normal forms and invariants, the foundation of the framework, are introduced in §3. The bifurcation and topology of equilibrium points for systems with a single uncontrollable mode are summarized in §4. Bifurcation control by state feedback is addressed in §5, such as the stability of stationary bifurcations. In §6 and §7, we review some related results on the feedback control with perturbation, and the bifurcation control using nonsmooth feedback.

2 Problem Formulation

The first question we have to answer is what do we mean by the terminology “bifurcation of control systems?” Let us review the bifurcation of a classical dynamic system $\dot{\xi} = f(\xi, \mu)$. The classical bifurcation theory studies the change of qualitative properties of the dynamical system as the parameter is varied. What are the qualitative properties? Some important properties include the topology of the equilibrium set, the stability, and the existence of periodic solutions. Therefore, if any of these properties is changed in a dynamical system as a parameter is varied, we say that the system exhibits a bifurcation. Typical bifurcations include saddle node, transcritical, pitchfork (both the topology of the equilibrium set and the stability are changed), and the Hopf bifurcation (a periodic solution is generated).

Let us follow the same way of thinking for control systems. The bifurcation theory of control systems should study the change of qualitative properties. What are the important qualitative properties for control systems? Some important properties of a control system include: the controllability, the stabilizability, and the topology of the equilibrium set. Therefore, the following problem of bifurcation for control systems is studied in this paper.

**Problem 1.** Bifurcation of control systems. It focuses on the change of qualitative properties of control systems such as controllability, stabilizability, and the topology of the equilibrium set.

So bifurcation of control systems may have a meaning that is different from bifurcation of dynamical systems without control. One may ask what is the relationship between the bifurcation of control systems and the classical bifurcation of dynamical systems? This question leads to another interesting topic that is addressed in this paper. We know that if a feedback $u = \alpha(\xi)$ is applied to a control system, it becomes a dynamical system without control (if feedforward is applied, it is a system with a parameter). It is shown in this paper that the bifurcation defined in Problem 1 is converted to a classical bifurcation in a closed-loop system. However, different feedbacks applied to
the same system may result in completely different classical bifurcations. This phenomenon leads to the second problem of research.

**Problem 2.** Bifurcation control using state feedback. The problem focuses on the feedback design to achieve the stability around a critical point, or to achieve the desired performance by qualitatively changing a classical bifurcation.

We know that qualitative properties such as the controllability and stabilizability of a control system are invariant under changes of coordinates and the regular feedback $u = \alpha(\xi) + \beta(\xi)v$ if $\beta(\xi) \neq 0$. It is also known that systems may become simpler if suitable changes of coordinates and feedbacks are applied. Although two systems may look different, it is possible that they actually are equivalent to each other under a change of coordinates and feedback. So, the two systems have the same qualitative properties and the same bifurcation. Therefore, if we can find simple control systems that are equivalent to a family of complicated control systems, the bifurcation found for these simple systems represents the bifurcation of all systems in the family. This approach significantly simplifies the problem. This viewpoint leads to the following question: what are the simplest systems under changes of coordinates and feedbacks for a given family of control systems? This is equivalent to the following problem of normal forms.

**Problem 3.** Given a family of nonlinear control systems, find a set of normal forms equivalent to systems in the family under changes of coordinates and feedbacks.

Although Problem 3 is raised here for the purpose of bifurcation analysis, it is an interesting question by itself. Actually when this problem was addressed in [15], it was introduced as an extension of feedback linearization and Brunovsky form. In [32], normal forms were also used in the study of feedforward systems. Problem 1 was addressed in [17] and [18] for systems with a single uncontrollable mode. Problem 2 was addressed in [19] and [20].

Consider the following control system with a parameter
\[
\dot{\xi} = f(\xi, \mu) + g(\xi, \mu)u, \quad f(0, 0) = 0, \quad (1)
\]
where $\xi \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input and $\mu$ is the parameter. We assume that the rank of $g(\xi, \mu)$ is $m$ at the point of interest. Unless it is otherwise specified, all vector fields and state feedbacks in this paper are $C^k$ for some $k > 0$ sufficiently large. System (1) is said to be linearly controllable at $(\xi, \mu) = (0, 0)$ if its linearization $(A, B)$,
\[
A = \frac{\partial f}{\partial \xi}(0, 0), \quad B = g(0, 0)
\]
is controllable. The origin $(\xi, \mu) = (0, 0)$ is called an equilibrium or equilibrium point of (1) because $\xi(t) = 0$ is a constant solution if $\mu = 0$ and $u = 0$. A constant solution may exist for other values of $(\xi, \mu, u)$. The equilibrium set is defined by
\[
E = \{(\xi, \mu) \mid \text{there exists } u_0 \in \mathbb{R} \text{ such that } f(\xi, \mu) + g(\xi, \mu)u_0 = 0\}.
\]
A point in $E$ is called an equilibrium or equilibrium point. Feedbacks are not involved in this definition. If the control input $u$ is substituted by a feedback $u = u(\xi, \mu)$, a closed-loop equilibrium, $(\xi_0, \mu_0)$, is defined by $f(\xi_0, \mu_0) + g(\xi_0, \mu_0)u(\xi_0, \mu_0) = 0$. The set of all closed-loop equilibria is

$$E_c = \{(\xi, \mu) | f(\xi, \mu) + g(\xi, \mu)u(\xi, \mu) = 0\}.$$  

The concept of the equilibrium set plays an important role in this paper. It is known that the closed-loop equilibrium set $E_c$, in general, is changed if the feedback is varied. However, the set $E_c$ under any state feedback must be a subset of $E$. Therefore, the set $E$ consists of all possible closed-loop equilibria.

It is assumed throughout this paper that there exists a single uncontrollable mode (denoted by $\lambda$) in the linearization. The dimension of the state space is at least two ($n \geq 2$). The dimension of the control input is one. If $\lambda \neq 0$, the sign of $\lambda$ determines the stabilizability of the uncontrollable dynamics. Therefore, a small variation of $\mu$ does not change the stability, i.e. there is no stationary bifurcation at $\mu = 0$. If $\lambda = 0$, the stability of the system depends on the value of the parameter. Several kinds of bifurcations occur in the performance. Thus, we focus on systems with $\lambda = 0$ in the following sections.

### 3 Normal Form and Invariants

In this section, Problem 3 is addressed. While we focus on the results from [15], [16], [17], [18], and [22], we would like to point out that some concepts from these papers were generalized to a larger family of systems such as discrete-time systems ([2]), systems with multiple uncontrollable modes ([9]) and normal forms of higher degree terms ([21] and [24]).

In the dissertation of Poincaré, normal forms were derived for dynamical systems without a control input. Poincaré’s idea is to simplify the linear part of a system first, using a linear change of coordinates. Then, the quadratic terms in the system are simplified, using a quadratic change of coordinates, then cubic terms, and so on. For control systems, we will use a similar idea. However, the normal form is different. Someone may ask: is it necessary to derive a different normal form for control systems, rather than using the Poincaré normal form? The answer is yes. In fact, even for a linear control system $\dot{\xi} = A\xi + Bu$, the controller normal form is more useful than the diagonal form of $A$ in the feedback design. For a nonlinear affine control system $\dot{\xi} = f(\xi) + g(\xi)u$, it has two vector fields $f(\xi)$ and $g(\xi)$. Therefore, the normal form of a control system requires the simplification of both $f$ and $g$ simultaneously. The simplification of $f$ does not necessarily result in a simple form for $g$. Furthermore, the transformation group of control systems consists of changes of coordinates and feedbacks. This is different from the Poincaré normal form of dynamical systems where feedbacks are not considered.
3.1 Normal form

Consider system (1). If the system is linearly controllable at the origin, the system is linearly controllable for all equilibrium points in $E$ around $(\xi, \mu) = (0, 0)$. Therefore, Problem 1 in §2 is interesting if (1) is not linearly controllable at the origin.

**Assumption 1:** System (1) has one uncontrollable mode, $\lambda = 0$, i.e.
\[
\text{rank} \left( \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \right) = n - 1, \lambda = 0.
\]  

The first step of finding normal forms is to simplify the linear part in its Taylor expansion. A linear transformation is the following linear change of coordinates and linear feedback
\[
\begin{align*}
\tilde{\xi} &= T_0 \mu + T_1 \xi \\
\tilde{u} &= a_0 \mu + a_1 \xi + \beta u
\end{align*}
\]  

where $T_1$ is invertible and $\beta \neq 0$. Once the linear part of the system is transformed into its normal form, a quadratic transformation in the form of
\[
\begin{align*}
\tilde{\xi} &= \xi + \phi^2(\mu, \xi) \\
\tilde{u} &= u + \alpha^2(\mu, \xi) + \beta^1(\mu, \xi)u
\end{align*}
\]  

The superscripts of $\phi^2(\mu, \xi)$, $\alpha^2(\mu, \xi)$, and $\beta^1(\mu, \xi)$ denote that the functions are homogeneous polynomials of second and first degree in $(\mu, \xi)$. Similar superscript is also applied to other vector fields and functions. In (4), the new and old variables agree at the linear level. This is important because such a transformation leaves the linear part of a system unchanged. So, it does not change the linear normal form while simplifying the quadratic part. The linear and quadratic normal forms are summarized in Theorem 1. In the normal form, the new variable has two components $\tilde{\xi} = [z \ x]^T$.

**Theorem 1.** ([18]) Consider system (1) satisfying Assumption 1. Then, there exists a linear and quadratic transformation that transforms the system into one of the following normal forms.

(i) **Double-zero uncontrollable mode.**
\[
\begin{align*}
\dot{z} &= \mu + \sum_{i=1}^{n-1} \gamma_{i} x \ x_i^2 + \gamma_{zx} z x_1 + \gamma_{z\mu} \mu x_1 + \gamma_{zz} z^2 + O(z, x, \mu, u)^3 \\
\dot{x} &= A_2 x + B_2 u + f_z^2(x) + O(z, x, \mu, u)^3.
\end{align*}
\]  

(ii) **Simple-zero uncontrollable modes.**
\[
\begin{align*}
\dot{z} &= \sum_{i=1}^{n-1} \gamma_{i} x \ x_i^2 + \\
&\quad + \gamma_{\mu} \mu^2 + \gamma_{zx} z x_1 + \gamma_{z\mu} \mu x_1 + \gamma_{zz} z^2 + O(z, x, \mu, u)^3 \\
\dot{x} &= A_2 x + B_2 u + f_z^2(x) + O(z, x, \mu, u)^3.
\end{align*}
\]
In (5) and (6), $\tilde{f}^{[2]}(x)$ is from [15], and $(A_2, B_2)$ is in Brunovsky form.

\[
\tilde{f}_i^{[2]}(x) = \sum_{j=i+2}^{n-1} a_{ij} x_j^2, \quad 1 \leq i \leq n - 3,
\]

\[
\tilde{f}_i^{[2]}(x) = 0, \quad i = n - 2, n - 1.
\]

\[
A_2 = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\in (n-1) \times (n-1)
\]

\[
B_2 = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

3.2 Invariants

In the matrix theory, the normal form of a matrix is the Jordan canonical form. Its invariants are eigenvalues. The eigenvalues are invariant under matrix transformations. In dynamical systems, eigenvalues are used to characterize the qualitative properties (properties that are invariant under change of coordinates) such as stability. As for nonlinear systems without control, the linear invariants are eigenvalues and the nonlinear invariants are the homogeneous resonant terms in the Poincaré normal form. It was proved by Poincaré that the resonant terms in his nonlinear normal form are invariant under homogenous changes of coordinates.

We know that qualitative properties of control systems, such as the topology of the equilibrium set and the controllability, are invariant under a change of coordinates and feedback. It is expected to characterize these properties by invariant numbers of control systems. In the linearization of a control system, the uncontrollable mode and the controllability index are invariant. What are the invariants in the nonlinear terms of a control system? The following result from [18] answers this question. For the quadratic invariants, we assume that a system is already transformed into its linear normal form (7) or (8).

\[
\dot{z} = \mu + f_1(z, x, \mu) + g_1(z, x, \mu)u
\]

\[
\dot{x} = A_2x + B_2u + f_2(z, x, \mu) + g_2(z, x, \mu)u,
\]

\[
\dot{z} = f_1(z, x, \mu) + g_1(z, x, \mu)u
\]

\[
\dot{x} = A_2x + B_2u + f_2(z, x, \mu) + g_2(z, x, \mu)u.
\]

The nonlinear terms $f_i$ and $g_i$, for $i = 1, 2$, are not necessarily in the quadratic normal form. If a quadratic transformation is applied to (7) or (8), the non-
linear vector fields are changed. However, the following numbers obtained from the nonlinear system are left invariant.

<table>
<thead>
<tr>
<th>Invariants</th>
<th>In Normal Form</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2}C_x A_t^{-1} <a href="0,0,0">ad_{f_e}^r(g_e), ad_{f_e}^{n-1}(g_e)</a>$</td>
<td>$a_t$</td>
<td>$n-r$</td>
</tr>
<tr>
<td>$\frac{1}{2}C_z <a href="0,0,0">ad_{f_e}^r(g_e), ad_{f_e}^{n-1}(g_e)</a>$</td>
<td>$\gamma_{x_{n-r} x_{n-r}}$</td>
<td>$1 \leq r \leq n-1$</td>
</tr>
<tr>
<td>$(-1)^{n-1} C_z <a href="0,0,0">X_{\mu}, ad_{f_e}^{n-1}(g_e)</a>$</td>
<td>$\gamma_{x_{\mu}}$</td>
<td>(7)</td>
</tr>
<tr>
<td>$(-1)^{n-1} C_z <a href="0,0,0">X_{\mu}, ad_{f_e}^{n-1}(g_e)</a>$</td>
<td>$\gamma_{x_{\mu}}$</td>
<td>(8)</td>
</tr>
<tr>
<td>$\frac{1}{2} C_z ad_{X_{\mu}}^2 (f)</td>
<td>_{(0,0,0)}$</td>
<td>$\gamma_{x_{\mu} x_{\mu}}$</td>
</tr>
<tr>
<td>$C_z ad_{X_{\mu}} ad_{X_{\mu}} (f)</td>
<td>_{(0,0,0)}$</td>
<td>$\gamma_{x_{\mu} x_{\mu}}$</td>
</tr>
</tbody>
</table>

In (9), $f_e$ (or $g_e$) is the extended vector field including $f$ (or $g$) in (7) or (8) and the parameter equation $\dot{\mu} = 0$. The notations $C_z, C_x, X_z$ and $X_{\mu}$ are the following row and column vectors in $\mathbb{R}^{n+1}$,

$$
C_z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad C_x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix},
X_z = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}^T,
X_{\mu} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T.
$$

The following theorem implies that the quadratic invariants characterize the quadratic part of a system.

**Theorem 2.** ([18]) Given a system satisfying Assumption 1. Suppose its linearization is (7) or (8).

(i) Quadratic transformations defined by (4) do not change the value of the quadratic invariants.

(ii) The quadratic invariants of normal form (5) or (6) are the corresponding coefficients of the quadratic terms listed in (9).

(iii) Two systems of the form (7) (or (8)) are equivalent under quadratic transformations if and only if they have the same quadratic invariants.

Cubic invariants and cubic normal forms were derived in several publications such as [21] and [24]. However, we do not review these results here. In the following, a set of special invariants are identified, which are not only easy to compute but also critical for the bifurcation analysis performed in the sections that follow.

### 3.3 Resonant terms

The Poincaré normal form consists of resonant terms. The coefficients of resonant terms are left invariant under homogeneous transformations. For control systems, there is a set of homogeneous polynomials that are invariant...
under homogeneous transformations. The coefficients of resonant terms form a subset of the invariants in (9). Furthermore, the resonant terms are critical to the feedback control of bifurcations.

Given a system in the form of (7) or (8). The homogeneous parts of degree \(d\) in \(f_i\) and \(g_i\) of (7) and (8) are denoted by \(f_i^{[d]}\) and \(g_i^{[d]}\). A homogeneous transformation of degree \([d]\) is defined by a change of coordinates and state feedback in the following form

\[
\begin{align*}
    z &= \bar{z} + \phi_i^{[d]}(\bar{z}, \bar{x}, \mu), \quad \bar{x} = \bar{x} + \phi_2^{[d]}(\bar{z}, \bar{x}, \mu) \\
    u &= \bar{u} + \alpha_1^{[d]}(\bar{z}, \bar{x}, \mu) + \beta_1^{[d-1]}(\bar{z}, \bar{x}, \mu)\bar{u}
\end{align*}
\]

where \(\bar{z}\) and \(\bar{x}\) are the new coordinates, \(\bar{u}\) is the new control input introduced by the regular feedback.

**Definition 1.** Consider (7) or (8). A homogeneous term in \(f_1^{[d]}(z, x, \mu)\) or \(g_1^{[d-1]}(z, x, \mu)u\) is called a resonant term if transformations defined by (11) leave the coefficient of the term invariant.

For instance, if (11) is applied to (7), it can be proved that the term \(z^2\) in \(f_1^{[2]}\) of the resulting system has the same coefficient as the term \(z^2\) in \(f_1^{[2]}\) of (7). So, \(z^2\) in \(f_1^{[2]}\) is resonant. Notice that the resonant terms of control systems are different from the resonant terms in the classical theory of dynamical systems. A resonant term in Definition 1 is invariant under both changes of coordinates and state feedback. This is different from the theory of classical dynamic systems that does not deal with feedback. Define

\[
\begin{align*}
    R_1^{[d]}(\mu, z, x_1) &= f_1^{[d]}(\mu, z, x)|_{x_2=x_3=\ldots=x_{n-1}=0}, \\
    R_1^{[d]}(z, x_1) &= R(z, x_1, 0)
\end{align*}
\]

where \(f_1^{[d]}(\mu, z, x)\) is the homogeneous vector field of degree \(d\) from the Taylor expansion of \(f_1(\mu, z, x)\) in (7) and (8).

**Theorem 3.** ([18]) In (7), all terms of \(R_1^{[d]}(z, x_1)\) are resonant. In (8), all terms of \(R_1^{[d]}(\mu, z, x_1)\) are resonant.

The coefficients of quadratic resonant terms are part of the invariants in (9). For example, the coefficient of \(z^2\) in \(R_1^{[2]}\) equals the invariant \(\gamma_{zz}\) in (9). The quadratic functions of resonant terms \(R_1^{[2]}(\mu, z, x_1)\) and \(R_1^{[2]}(z, x_1)\) are also denoted by \(Q(\mu, z, x_1)\) and \(Q_1(z, x_1)\), i.e., \(Q(\mu, z, x_1) = R_1^{[2]}(\mu, z, x_1)\) and \(Q_1(z, x_1) = R_1^{[2]}(z, x_1)\). The coefficients in \(R_1^{[2]}\) and \(R_1^{[2]}\) form two matrices,

\[
Q = \begin{bmatrix}
\gamma_{\mu\mu} & \frac{\gamma_{\mu z}}{2} & \frac{\gamma_{\mu x_1}}{2} \\
\frac{\gamma_{\mu z}}{2} & \gamma_{zz} & \frac{\gamma_{z x_1}}{2} \\
\frac{\gamma_{\mu x_1}}{2} & \frac{\gamma_{z x_1}}{2} & \gamma_{x_1 x_1}
\end{bmatrix}, \quad Q_1 = \begin{bmatrix}
\gamma_{zz} & \frac{\gamma_{z x_1}}{2} \\
\frac{\gamma_{z x_1}}{2} & \gamma_{x_1 x_1}
\end{bmatrix}.
\]

The matrices satisfy

\[
Q(\mu, z, x_1) = [\mu \ z \ x_1] Q [\mu \ z \ x_1]^T, \quad Q_1(z, x_1) = [z \ x_1] Q_1 [z \ x_1]^T.
\]
4 Bifurcation of Control Systems

In §3, normal forms of systems satisfying Assumption 1 are derived. The bifurcation of the normal form represents the bifurcation of all systems satisfying Assumption 1. In this section, we study the bifurcation of the normal form (5) and the bifurcation of the controllability. Later in §5, we will prove that these bifurcations correspond to classical stationary bifurcations of closed-loop systems. The relationship between the bifurcation of control systems and the classical stationary bifurcations will also be discussed in §5.

4.1 Systems with a double-zero uncontrollable mode

Consider (7). It can be transformed into the normal form (5) by a suitable quadratic transformation. In [18], we focus on the resonant terms in the normal form (5) rather than studying arbitrary nonlinear terms in (7). It significantly simplifies the analysis.

**Theorem 4.** ([18]) Let a system in the form of (7) be given.

(i) Its equilibrium set satisfies

\[
\begin{align*}
\mu &= -Q_1(z, x_1) \\
&= -\gamma_{zx_1}z^2 - \gamma_{zzz}z^2 + O(z, x_1)^3 \\
x_i &= O(z, x_1)^2 \\
&\quad 2 \leq i \leq n - 1.
\end{align*}
\]

(ii) There exists a function \(c(z, x_1)\) in the following form

\[
c(z, x_1) = \gamma_{zx_1}z + 2\gamma_{zx_1}x_1 + O(z, x_1)^2
\]

such that the system is linearly controllable at a point \((z, x, \mu) \in E\) if and only if \(c(z, x_1) \neq 0\).

Equation (15) is a quadratic approximation of the equilibrium set \(E\) (projected to \(\mu z x_1\)-space). If \(Q_1\) is sign definite (or equivalently \(\det(Q_1) > 0\)), \(E\) is approximately a paraboloid. If \(Q_1\) is not sign definite but it has full rank (or equivalently \(\det(Q_1) < 0\)), then \(E\) is approximately a saddle. If we define

\[E_{\mu_0} = \{(x, \mu) \in E|\mu = \mu_0\},\]

the topology of \(E_{\mu}\) changes as \(\mu\) passes through zero. If \(E\) is approximately a paraboloid, \(E_{\mu}\) is empty for the values of \(\mu\) on one side of zero and it is a closed curve if \(\mu\) is on the other side of \(\mu = 0\). If \(E\) is approximately a saddle, then \(E_0\) is approximately a set with two lines which meet at the origin. It is a connected set. However, \(E_{\mu}\) is approximately a hyperbola for \(\mu \neq 0\), which is not a connected set. Examples of a paraboloid and saddle are shown in Figure 1. The system is uncontrollable at the intersection of \(c(z, x_1) = 0\) and \(E\). The uncontrollable equilibrium points are shown in the figure by the curve on the surface \(E\).
4.2 Systems with simple-zero uncontrollable modes

For system (8), the set $E$ is approximated by the resonant terms $Q(\mu, z, x_1)$ as proved in the following theorem.

**Theorem 5.** ([18]) Given a system (8).

(i) If $Q$ is sign definite, the origin is an isolated equilibrium point, i.e. $E = \{(0, 0, 0)\}$.

(ii) If $Q$ is indefinite and if $Q$ has full rank, its equilibrium set satisfies

$$
\begin{align*}
\left[ \begin{array}{c}
\mu \\
z \\
x_1
\end{array} \right] Q \left[ \begin{array}{c}
\mu \\
z \\
x_1
\end{array} \right] + O(\mu, z, x_1)^3 &= 0, \\
x_i &= O(\mu, z, x_1)^2 \\
&\quad 2 \leq i \leq n - 1.
\end{align*}
$$

(iii) In the case of (ii), there exists a function $c(\mu, z, x_1)$ in the following form

$$
c(\mu, z, x_1) = 2 \left[ \begin{array}{ccc}
0 & 0 & 1
\end{array} \right] Q \left[ \begin{array}{c}
\mu \\
z \\
x_1
\end{array} \right] + O(\mu, z, x_1)^2
$$

such that the system is linearly controllable at a point $(\mu, z, x) \in E$ if and only if $c(\mu, z, x_1) \neq 0$.

In (i), the system has an equilibrium point only if $\mu = 0$. When $\mu \neq 0$, the system has no equilibrium point in a neighborhood of the origin. In (ii) and (iii), the equilibrium set $E$ is approximately a cone. The uncontrollable equilibrium points are approximately the intersection of the cone with a plane. The intersection has two generic cases, it may consist of two lines (as shown in Figure 2), or it consists of a single point that is the origin.

To summarize, the topology of the equilibrium set $E$ and the set of uncontrollable points are completely determined by the resonant terms $Q$ or $Q_1$ of the system. The relationship between $Q$, $Q_1$, and $E$ is summarized in the following table.
Table 1. The classification of equilibrium sets

<table>
<thead>
<tr>
<th>Uncontrollable Mode</th>
<th>Condition</th>
<th>Equilibrium Set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double-zero</td>
<td>$\text{det}(Q_1) &gt; 0$</td>
<td>paraboloid</td>
</tr>
<tr>
<td></td>
<td>$\text{det}(Q_1) &lt; 0$</td>
<td>saddle</td>
</tr>
<tr>
<td>Simple-zero</td>
<td>$Q$ is sign definite</td>
<td>single point</td>
</tr>
<tr>
<td></td>
<td>$Q$ is indefinite, $\text{det}(Q) \neq 0$</td>
<td>cone</td>
</tr>
</tbody>
</table>

5 Bifurcation Control Using State Feedback

It has been observed in engineering and scientific applications that a feedback is able to change not only the stability of a bifurcation but also its type. The same system may exhibit more than one type of bifurcations, depending on the selection of the feedback. In [19], all possible bifurcations with quadratic and cubic degeneracies that can be generated by (7) and (8) are found. It is proved that the bifurcations are completely determined by the feedback and the resonant terms. Actually, the bifurcation and its stability is closely related to the equilibrium set $E$, which is summarized in §4.

\[ u = a_\mu \mu + a_z z + \sum_{i=1}^{n-1} a_i x_i + a^{[2]}(\mu, z, x) + O(z, x, \mu)^3. \] (19)

To guarantee that the system is operated around the origin, we assume,

**Assumption 2**: State feedback (19) places the controllable poles in the left half plane, i.e. the eigenvalues of the matrix $A_2 + B_2 \left[ a_1 \ a_2 \ \cdots \ a_{n-1} \right]$ are all in the left half plane.
5.1 Bifurcations with quadratic degeneracy

For a closed-loop system under feedback (19), the input \( u \) is replaced by (19). For the closed-loop system, the equilibrium set is denoted by \( E_c \), which consists of all \((\mu, z, x)\) so that the vector field with feedback equals zero. A definition of \( E_c \) is introduced in §2. Given (8), it is easy to check that \( E_c \), the closed-loop equilibrium set, is the intersection of \( E \) and the set \( u(\mu, z, x) = 0 \). In a neighborhood of the origin, the intersection is approximately the intersection of the plane

\[
a_{\mu} \mu + a_z z + \sum_{i=1}^{n-1} a_i x_i = 0
\]

with \( E \). We know that \( E \) is either a single point or a cone. So, the intersection between (20) and \( E \) is either a single point or two lines passing through the origin. The later case obviously implies a transcritical bifurcation. It is proved to be true and the result is summarized in the following theorem. The following 2 × 2 matrix is critical, which is determined by the invariants \( Q \) and the linear coefficients of the feedback,

\[
\tilde{Q} = \begin{bmatrix} a_1 & 0 \\ 0 & a_1 - a_z \end{bmatrix} Q \begin{bmatrix} a_1 & 0 \\ 0 & a_1 \\ -a_\mu & -a_z \end{bmatrix}.
\]

**Theorem 6.** Consider a closed-loop system (8)-(19) satisfying Assumption 2. Suppose

\[
Q_1(a_1, -a_z) \neq 0.
\]

(i) If \( \tilde{Q} \) is sign definite, then \((z, x, \mu) = (0, 0, 0)\) is an isolated equilibrium point of the closed-loop system. It is unstable.

(ii) If \( \tilde{Q} \) is indefinite with full rank, then the closed-loop system has a transcritical bifurcation around the origin.

(iii) Assume that the feedback satisfies the condition in (ii). Given any \((z, x, \mu) \in E_c \) in a neighborhood of the origin, it is locally asymptotically stable if

\[
\begin{bmatrix} 0 & a_1 - a_z \end{bmatrix} Q \begin{bmatrix} \mu & z & x_1 \end{bmatrix}^T > 0
\]

The system is unstable if

\[
\begin{bmatrix} 0 & a_1 - a_z \end{bmatrix} Q \begin{bmatrix} \mu & z & x_1 \end{bmatrix}^T < 0.
\]

An example of engine compressors is given in [19]. The compressor model is based on the 3D Moore-Greitzer model. A transcritical bifurcation occurs in this model. To achieve the stability on a given branch of the equilibrium, the controllers are derived on the basis of Theorem 6.

For system (7), the set \( E \) is either a paraboloid or a saddle. The intersection between (20) and \( E \) is a parabola. Therefore, it indicates a saddle node bifurcation. This is proved to be true (Table 2). The stability of an equilibrium point is determined by the sign of \( Q_1(z, x_1)z \). Details can be found in [19].
5.2 Bifurcations with cubic degeneracy

On the center manifold of (8)-(19), condition (22) implies that the quadratic terms of the reduced system dominate the local performance. However, if $Q_1(a_1, -a_z) = 0$, the performance of the reduced system on the center manifold is affected by cubic terms in the system. It is proved in [19] that the pitchfork bifurcation occurs and the stability is closely related to the quadratic feedback as well as cubic resonant terms in $R[3](\mu, z, x_1)$. In fact, only the cubic resonant terms of $z$ and $x_1$ are important. So, we define

$$C(z, x_1) = R[3]_1(0, z, x_1) = f[3]_1(\mu, z, x)|_{x_2=x_3=\cdots=x_{n-1}=\mu=0},$$

(25)

where $f[3]_1$ is the cubic part in the uncontrollable equation of (6). The coefficients in the quadratic feedback $x[2]_z(\mu, z, x)$ are denoted by $a_{zz}$, $a_{z\mu}$, $a_{zz_1}$, $a_{z_1\mu}$, etc. The following quadratic function from $x[2]_z$ in (19) is useful,

$$x[2]_{zz_1}(z, x_1) = a_{zz}z^2 + a_{zz_1}zx_1 + a_{x_1x_1}x^2_1,$$

(26)

i.e. $x[2]_{zz_1}$ is the restriction of $x[2]_z$ to the $zz_1$-plane. The following number, determined by invariants and feedback, is critical for bifurcation control,

$$D = a_1C(a_1, -a_z) + (2a_z\gamma_{x_1x_1} - a_1\gamma_{zz_1})x[2]_{zz_1}(a_1, -a_z)$$

(27)

where $\gamma_{x_1x_1}$ and $\gamma_{zz_1}$ are quadratic resonant coefficients in (6).

**Theorem 7.** Consider a closed-loop system (6)-(19) satisfying Assumption 2. Suppose

$$Q_1(a_1, -a_z) = 0.$$  

(28)

(i) The closed-loop system has a pitchfork bifurcation at the origin if

$$D \neq 0,$$

$$\begin{bmatrix} 0 & a_1 & -a_z \end{bmatrix} Q \begin{bmatrix} a_1 & 0 & -a_\mu \end{bmatrix}^T \neq 0.$$  

(29)

(ii) The pitchfork bifurcation is supercritical if $D < 0$, and subcritical if $D > 0$.

Suppose that a control system has a transcritical bifurcation. An interesting conclusion of Theorem 7 is that a feedback is capable of changing the transcritical bifurcation into a pitchfork bifurcation, a different bifurcation type. For the normal form (5), the change is even more dramatic. The condition $Q_1(a_1, -a_z) \neq 0$ implies a saddle node bifurcation. On one side of $\mu = 0$, the system has no equilibrium point, and therefore cannot be stabilized. However, it is proved in [19] that a feedback satisfying $Q_1(a_1, -a_z) = 0$ removes the saddle node bifurcation. The resulting system has no bifurcation if $D \neq 0$. Locally the system has a unique equilibrium point for every value of $\mu$ in a neighborhood of $\mu = 0$. Furthermore, if $D < 0$, all the equilibrium points around the origin are stable. Notice that a linear controller cannot
remove the bifurcation because the system is not linearly controllable. The stabilization is done by the quadratic part of the feedback. The relationship between the feedback, the resonant terms and the bifurcation is summarized in the following table.

Table 2. The bifurcations under state feedback

<table>
<thead>
<tr>
<th>System</th>
<th>Condition</th>
<th>Bifurcation</th>
<th>Stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple-zero Uncontrollable Mode</td>
<td>( Q_1(a_1, -a_z) \neq 0 )</td>
<td>det((\bar{Q})) &gt; 0</td>
<td>The origin is an isolated equilibrium point</td>
</tr>
<tr>
<td></td>
<td>det((\bar{Q})) &lt; 0</td>
<td>Transcritical bifurcation</td>
<td>The system is unstable</td>
</tr>
<tr>
<td>Double-zero Uncontrollable Mode</td>
<td>( Q_1(a_1, -a_z) = 0 )</td>
<td>( D \neq 0 ) ( \bar{Q}(a_1,0,-a_\mu) \neq 0 )</td>
<td>Pitchfork bifurcation</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Supercritical if ( D &lt; 0 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Subcritical if ( D &gt; 0 )</td>
</tr>
</tbody>
</table>

In the table, \( Q_1 \) is defined by (14), \( \bar{Q} \) is defined by (21), \( D \) is defined by (27), and \( \bar{Q} \) is a function with three variables defined by

\[
\bar{Q}(a,b,c) = \begin{bmatrix} 0 & a_1 & -a_z \end{bmatrix} Q \begin{bmatrix} a & b & c \end{bmatrix}^T.
\]

6 The Cusp Bifurcation and Hysteresis

From Table 2, system (5) has no bifurcation if \( Q_1(a_1, -a_z) = 0 \) and \( D \neq 0 \). However, this statement is based on the assumption that \( \mu \) is the only parameter in the system. If the system has other parameters, it may exhibit rich bifurcation phenomena. For example, it was proved in [19] that the bifurcation has qualitative change if the feedback is perturbed by a constant. In other words, if the feedback is not zero at the origin, it might result in a fundamental change in the bifurcations of the system. In this section, we use (5) as an example to illustrate the phenomenon.

Given a system in the form of (5), suppose that

\[ Q_1(a_1, -a_z) = 0, \quad D \neq 0. \] (30)

Suppose that the feedback equals \( \nu \) at the origin, i.e.

\[ u = \nu + a_\mu \mu + a_z z + \sum_{i=1}^{n-1} a_i x_i + \alpha^2(\mu, z, x) + O(z, x, \mu)^3. \] (31)
It was proved in [19] that the center manifold of (5)-(31) and the reduced system satisfy
\[
\begin{align*}
x_1 &= -a_z z - \frac{a_1 a_\mu}{a_1^2} \mu - \frac{1}{a_1} \nu + O(z, \mu)^2, \\
x_2 &= -a_z \mu + O(z, \mu)^2, \\
x_i &= O(z, \mu)^2 \quad \text{for} \quad i \geq 3,
\end{align*}
\]
(32)
\[
\dot{z} = f_c(z, \mu, \nu) = \nu_1 + \nu_2 z + D_2(\mu, \nu) z^2 + D_3 z^3 + \cdots,
\]
(33)
where \( D \) is defined by (27), and
\[
\begin{align*}
\nu_1 &= \mu + O(\mu, \nu)^2, \\
\nu_2 &= \left( 2 \gamma_{xx} a_z - \gamma_{x1} a_1 \right) \left( a_\mu - \frac{a_2 a_z}{a_1} - \frac{\gamma_{x1} \mu a_z}{a_1} \right) \mu + \\
&\quad + \frac{2 a_z \gamma_{x1} - a_1 \gamma_{x1} \nu}{a_1^2} + \cdots,
\end{align*}
\]
(34)
On the center manifold, the reduced system is (33). If \( \nu_1 \) and \( \nu_2 \) in (33) are free variables, the system exhibits a cusp bifurcation ([13]). The system has cubic degeneracy. The set of degenerate equilibrium points is a cusp. The figure of \( f_c \) for fixed values of \( (\nu_1, \nu_2) \) and the stability at the equilibrium are illustrated in the following figure.

Fig. 3. A cusp and phase portrait of \( f_c \). \( D_3(0) < 0 \) in (a). \( D_3(0) > 0 \) in (b).

From (34), \( \nu_i, i = 1, 2, \) depends on the value of \( (\nu, \mu) \). For any fixed value of \( \nu \), (34) represents a curve in the \( \nu_1 \nu_2 \)-plane. The bifurcation of (5)-(31) is determined by this curve and the cusp in Figure 3. If the curve has no intersection with the cusp, the cusp bifurcation diagram implies that there is no bifurcation around the origin (Figure 4a-b). If the curve meets with the cusp (Figure 4c-d), the bifurcation diagram is shown in Figure 5, which is called a hysteresis ([34]). We summarize the results in the following theorem.

**Theorem 8.** Consider (5)-(31) satisfying Assumption 2 and (30). Fix any value of \( \nu \) around \( \nu = 0 \), (5)-(31) has the following properties.
(i) If \((2a_2\gamma x_2z_1 - a_1\gamma x_1)D\nu > 0\), \((5)-(31)\) has no bifurcation around the origin and \(\mu = 0\). There exists a neighborhood of \((z, x, \mu) = (0, 0, 0)\) in which the system has a unique closed-loop equilibrium point for every value of \(\mu\). Furthermore, the system is stable if \(D < 0\), and the system is unstable if \(D > 0\).

(ii) If \((2a_2\gamma x_1z_1 - a_1\gamma x_1)D\nu < 0\), the system exhibits hysteresis (Figure 5). Its stability is determined by the value of \(D\) as shown in Figure 5.

In [19], perturbed feedback is applied to (6) also. If \(E\) is a cone, it is proved that the transcritical bifurcation splits to two saddle node bifurcations.
7 Other Related Issues

If a continuous but nonsmooth feedback is applied to the system, it is possible to stabilize a bifurcation that cannot be stabilized by smooth feedbacks. In [25], a transcritical bifurcation is converted to a “birdfoot” bifurcation, which does not exist for smooth systems. The birdfoot bifurcation looks similar to the pitchfork bifurcation. The equilibrium set has a nonsmooth corner at the origin. The system can be made supercritical. It is more stable than a transcritical bifurcation.

In [19], nonsmooth feedback is applied to a system with saddle-node bifurcation. Consider a normal form with double-zero uncontrollable mode. The nonsmooth feedback is

\[ a_z|z| + a_1x_1 + \cdots + a_{n-1}x_{n-1} + a_\mu x_\mu + O(z, x, \mu)^2. \]

Suppose \( a_z \) satisfies

\[ Q_1(a_1, -a_z) < 0, Q_1(a_1, a_z) > 0. \]

Then the closed-loop system has a unique equilibrium point for every value of \( \mu \) around the origin. The equilibrium point is stable if \( \mu \neq 0 \). So, the system is stable for both \( \mu > 0 \) and \( \mu < 0 \). It compares sharply against the saddle-node bifurcation, where only one side of \( \mu = 0 \) has equilibrium points.

In [20], the Hopf bifurcation is studied. The paper assumes that a control system has two uncontrollable modes at \( \mu = 0 \), both on the imaginary axis and conjugate to each other. Since the imaginary eigenvalues are uncontrollable, the Hopf bifurcation cannot be removed by pole placement. For a nonzero parameter \( \mu \), the system may exhibit periodic solutions. Since the periodic solution lies on the center manifold, the orientation of the center manifold at the origin indicates the orientation of the periodic solution around the origin. It is proved in [20] that the orientation of the center manifold at the origin is determined by the linear feedback. Necessary and sufficient conditions are derived for all orientations that can be achieved by feedbacks. The stability of the periodic solution in the Hopf bifurcation is determined by its quadratic and cubic invariants, and the quadratic feedback. Quadratic normal forms are derived in [20] for systems with two imaginary uncontrollable modes. Normal forms of higher degrees are derived in [24]. In [20], a family of feedbacks is derived to stabilize the periodic solution of the Hopf bifurcation. Since the stability depends on the quadratic center manifold, the control of a quadratic center manifold by nonlinear feedback is addressed in [20]. An explicit formula that bridges the quadratic center manifold and the feedback is derived. Similar work about center manifold control was first studied in [9].

All results introduced so far assume that the system has a parameter \( \mu \). However, a control system without a parameter could have a bifurcation too. This is different from the classical bifurcation theory. In [17], it was proved...
that the topology of the equilibrium set is changed even without a parameter. The control input $u$ changes the equilibrium set. Furthermore, control bifurcation was introduced in [24]. In this work, a family of smooth feedbacks on the equilibrium set is applied to the system. As the equilibrium point is changed, the stability of the feedback changes, thus the control exhibits a bifurcation. There is no explicit parameter in the system. However, the change of input value or the change of equilibrium points results in the qualitative change of the performance of the given feedback.

Normal forms and bifurcation control of discrete-time systems have been studied in several recent publications (see, for instance, [2] and [14]). In addition to bifurcations, normal forms and invariants have other applications. For example, in [32] and references therein, the authors addressed the problems of control system symmetry, and feedforward forms based on invariants and normal forms.

8 Conclusions

We believe that the normal form and its invariants represent the essential nonlinearity in a control system. Therefore, the improvement in understanding normal forms and invariants will lead to a better understanding of nonlinear performance of control systems. We know that bifurcations are purely nonlinear performance because the linear part of a system at a bifurcation point does not dominate even the local performance due to the imaginary or zero eigenvalues. Therefore, the stability of bifurcations has to be closely related to its core nonlinear terms in the system. The results in this paper prove that invariants play a key role in the characterization of the qualitative performance. The proofs of these results are made possible by the normal form.

The research on normal forms, invariants, bifurcations and their applications has been fruitful for the last few years. However, the research on this topic is still far from mature. Normal forms of multiple uncontrollable modes are one of our next research target. Recent work on this issue is reported in [24]. Normal forms of higher degree terms are also under investigation. The convergence of normal forms is still largely open. While the distribution of eigenvalues of a system without control determines the convergence of the Poincaré normal form, the eigenvalues in a controllable system can be changed by feedback. So, what is the key that determines the convergence of a control system normal form is a tough challenge that we face. Bifurcations and their control is still a young field that has a great potential in the future. In this review, the bifurcation of the topology of the equilibrium set is thoroughly studied. However, there are certainly many other qualitative properties that are important to control systems, such as stabilizability, observability, and zero dynamics. The bifurcations of these qualitative prop-
Normal forms and bifurcations of control systems

Properties are interesting problems for future research. Bifurcation and chaos of discrete control systems is another interesting area for future research.

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